COMBINATORICA 7 (3) (1987) 215—219

ON A PROBLEM OF ERDŐS CONCERNING PROPERTY B

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Received 9 January 1986

A family $\mathcal{F}$ of sets has property B if there exists a set S such that $S\cap F\neq\emptyset$ and $S\not\supset F$ for every $F\in\mathcal{F}$. $\mathcal{F}$ has property $B(s)$ if there exists a set S such that $0<|F\cap S|<s$ for every $F\in\mathcal{F}$. Denote by $m(n)$ (respectively $m(n, s)$) the size of a smallest family of n-element sets not having property B (respectively $B(s)$). P. Erdős has asked whether $m(n, s)\geqq m(s)$ for all $n\geqq s$. We show that, in general, this inequality does not hold.

A family $\mathcal{F}$ of sets is said to have property B if there exists a set S such that $S\cap F\neq\emptyset$ and $S\not\supset F$ for every $F\in\mathcal{F}$. $\mathcal{F}$ is said to have property $\mathrm{B}(s)$ if there exists a set S such that $0<|F\cap S|<s$ for every $F\in\mathcal{F}$. Let n and s be positive integers, $n\geqq s$. Denote by $m(n)$ (respectively $m(n, s)$) the size of a smallest family of n-element sets not having property B (respectively $\mathrm{B}(s)$). Note that $m(n, n)=m(n)$. A survey of known results concerning $m(n)$ and $m(n, s)$ is given in [2].

P. Erdős has asked whether $m(n, s)\geqq m(s)$? This is clearly so if $s=1$ since $m(n, 1)=m(1)=1$, and is known to be so for $s=2, 3$ (see [1]). The object of this note is to prove that the answer, in general, is no. We shall prove the following theorems:

Theorem 1. *For all* $s\geqq 1$,

$$m(2^{s-1}, s) \leqq 2^s-1. \tag{1}$$

Theorem 2. *For all* $s\geqq 1$ *and* $n\geqq 5\cdot 2^{2s-3}$

$$m(n, s) \leqq 2^{s+2}-2s-4. \tag{2}$$

It has been shown by Beck [4] that for every $\varepsilon>0$, $m(s)>s^{(1/3)-\varepsilon}2^s$, provided $s\geqq s_0(\varepsilon)$. It is clear that Theorem 2 and this result imply that $m(n, s)<m(s)$ for s large and n large compared to s. We remark that Spencer [5] has given a proof of Beck's theorem which is simpler than that given in [4], although it is based on similar ideas.

Proof of Theorem 1. Let V_s denote the s-dimensional vector space over $\mathbf{Z}_2$. By a hyperplane we mean an $(s-1)$-dimensional subspace of V_s; equivalently, the set of

AMS subject classification (1980): 05 A 05

points $(x_1, x_2, \ldots, x_s)$ satisfying an equation of the form $a_1x_1+a_2x_2+\ldots+a_sx_s=0$. Let $\mathscr{H}_s$ denote the set of all hyperplanes and let $\mathscr{V}_s=\{\bar{H}\colon H\in\mathscr{H}_s\}$, where $\bar{H}$ denotes, as usual, the complement of H. Observe that if H is given by $a_1x_1+a_2x_2+\ldots+a_sx_s=0$, then $\bar{H}$ is given by $a_1x_1+a_2x_2+\ldots+a_sx_s=1$. We show that $\mathscr{V}_s$ does not have property $\mathrm{B}(s)$. Let S be a set which meets each member of $\mathscr{V}_s$. It cannot occur that the subspace of V_s spanned by S has dimension $\leqq s-1$, because if this were the case S would lie in some hyperplane H and thus miss $\bar{H}$. Hence S contains s linearly independent points, $X_i=(x_{i1}, x_{i2}, \ldots, x_{is})$, $i=1, 2, \ldots, s$. Since $\det[x_{ij}]\neq 0$, the system

$$x_{i1}a_1+x_{i2}a_2+\ldots+x_{is}a_s = 1, \quad i = 1, 2, \ldots, s$$

has a (unique) solution $a_1, a_2, \ldots, a_s$. Then the hyperplane H given by $a_1x_1+a_2x_2+\ldots+a_sx_s=0$ does not contain any of the points X_i, $i=1, 2, \ldots, s$. Thus $|S\cap\bar{H}|\geqq s$. as required. Since $\mathscr{V}_s$ consists of 2^s-1 sets of size 2^{s-1}, (1) holds. ▮

In the proof of Theorem 2 we shall need to use the following two constructions:

The abutting construction: Let $\mathscr{F}_1=\{F_j^1\colon j=1, 2, \ldots, m\}$ and $\mathscr{F}_2=\{F_j^2\colon j=1, 2, \ldots, m\}$ be families of sets satisfying $\hat{\mathscr{F}}_1\cap\hat{\mathscr{F}}_2=\emptyset$. Here, and in what follows, we use $\hat{\mathscr{F}}$ to denote

$$\bigcup_{F\in\mathscr{F}} F.$$

The family $\{F_j^1\cup F_j^2\colon j=1, 2, \ldots, m\}$ is said to be obtained by abutting $\mathscr{F}_1$ to $\mathscr{F}_2$. The construction generalizes to the case of more than two families in the obvious way. If A is a set disjoint from $\hat{\mathscr{F}}$, the family $\{F\cup A\colon F\in\mathscr{F}\}$ is said to be obtained by abutting A to $\mathscr{F}$. We denote this family by $(\mathscr{F}, A)$.

Let $\mathscr{F}^*$ be obtained by abutting any number of copies of $\mathscr{F}$. It was pointed out in [1], and it is easy to prove, that if $\mathscr{F}$ does not have property $\mathrm{B}(s)$, neither does $\mathscr{F}^*$. We observe, in anticipation of the next construction, that if every t-subset of $\hat{\mathscr{F}}$ is contained in at least one member of $\mathscr{F}$ then every t-subset of $\hat{\mathscr{F}}^*$ is contained in some member of $\mathscr{F}^*$.

The tower construction: Let $1=l_1<l_2<\ldots<l_s$ be positive integers. Let n be a positive integer and write $n=q_il_i+r_i$, $0\leqq r_i<l_i$. Let t be the largest integer such that $r_t=0$. For $k=t, t+1, \ldots, s$ let $\mathscr{F}_k$ be a family of sets with the following properties:

(i) $|F|=l_k$ for each $F\in\mathscr{F}_k$;
(ii) $\mathscr{F}_k$ does not have property $\mathrm{B}(k)$;
(iii) each $(k-1)$-subset of $\hat{\mathscr{F}}_k$ occurs in at least one member of $\mathscr{F}_k$.

Let $\mathscr{F}_k^*$ be the family obtained by abutting q_k copies of $\mathscr{F}_k$. Then each member of $\mathscr{F}_k^*$ is of size q_kl_k and, by the remarks made in connection with the abutting construction, $\mathscr{F}_k^*$ has properties (ii) and (iii).

Suppose that

(iv) $$|\hat{\mathscr{F}}_t^*| \leqq |\hat{\mathscr{F}}_{t+1}^*| \leqq\ldots\leqq |\hat{\mathscr{F}}_s^*|.$$

It is clear that we may construct the families $\mathscr{F}_k^*$ so that

(iv)′ $$\hat{\mathscr{F}}_t^* \subseteq \hat{\mathscr{F}}_{t+1}^* \subseteq\ldots\subseteq \hat{\mathscr{F}}_s^*.$$

Suppose further that

$$\text{(v)} \qquad |\hat{\mathscr{F}}^*_{s-1}|+r_{s-1}+r_{s-2}+\ldots+r_{t+1} \leqq |\hat{\mathscr{F}}^*_s|.$$

Then we may choose sets $A_{s-1}, A_{s-2}, \ldots, A_{t+1}$, which are pairwise disjoint, and which satisfy $A_k \subset \hat{\mathscr{F}}^*_s - \hat{\mathscr{F}}^*_{s-1}$ and $|A_k|=r_k$ for $k=s-1, s-2, \ldots, t+1$. Let A_s be a set of size r_s which is disjoint from $\hat{\mathscr{F}}^*_s$ and let $\mathscr{F}=\mathscr{F}^*_t \cup (\mathscr{F}^*_{t+1}, A_{t+1}) \cup \ldots \ldots \cup (\mathscr{F}^*_s, A_s)$. $\mathscr{F}$ is a family of n-element sets and $|\mathscr{F}|=|\mathscr{F}^*_s|+|\mathscr{F}^*_{s-1}|+\ldots+|\mathscr{F}^*_t|$.

We claim that $\mathscr{F}$ does not have property B(s). If $t=s$ then $\mathscr{F}=\mathscr{F}^*_s$ and everything reduces to the abutting construction. Hence suppose $t<s$. Let S be a set which meets each member of $\mathscr{F}$. We need to show that $|S \cap F| \geqq s$ for some $F \in \mathscr{F}$. If $S \cap A_s = \emptyset$, this is clearly the case since $\mathscr{F}^*_s$ does not have property B(s). Thus suppose $\exists a_s \in S \cap A_s$. If $S \cap A_{s-1} = \emptyset$, S must meet each member of $\mathscr{F}^*_{s-1}$ and, since $\mathscr{F}^*_{s-1}$ does not have property B$(s-1)$, must contain $s-1$ elements of some member of $\mathscr{F}^*_{s-1}$. In view of (iv)$'$ these $s-1$ elements must occur in some member of $\mathscr{F}^*_s$. This, and the fact that a_s is in every member of $(\mathscr{F}^*_s, A_s)$, yields the desired conclusion. We may therefore suppose $\exists a_{s-1} \in S \cap A_{s-1}$. If $S \cap A_{s-2} = \emptyset$ we find, by repetition of the above argument, that S must contain $s-2$ elements of some member of $\mathscr{F}^*_{s-2}$. These elements, together with a_{s-1}, occur in some member of $\mathscr{F}^*_s$ and S thus contains s elements of some member of $(\mathscr{F}^*_s, A_s)$. It is clear that repetition of this argument leads to the conclusion that $\mathscr{F}$ does not have property B(s).

Proof of Theorem 2. For $1 \leqq k \leqq s$, let $\mathscr{V}_k$ be the family constructed in the proof of Theorem 1. For each $H \in \mathscr{H}_k$ choose an element $h \in \overline{H}$ and let H' be the set obtained from H by replacing 0 (the zero vector) by h. Let $\mathscr{V}_k' = \{H' \colon H \in \mathscr{H}_k\}$. We now show that the choice $l_k = 2^{k-1}$ and $\mathscr{F}_k = \mathscr{V}_k \cup \mathscr{V}_k'$ in the tower construction is a legitimate one in that conditions (i)—(v) are satisfied.

$\mathscr{F}_k$ consists of $2^{k+1}-2$ sets, each of size 2^{k-1}, and $|\hat{\mathscr{F}}_k| = 2^k - 1$. It is clear that $\mathscr{F}_k$ does not have property B(k) (because $\mathscr{V}_k$ does not) and that every $(k-1)$-subset of $\hat{\mathscr{F}}_k$ occurs in some member of $\mathscr{F}_k$ (in fact, in some member of $\mathscr{V}_k'$). Thus, (i), (ii) and (iii) are satisfied.

We verify that (iv) holds; that is, $|\hat{\mathscr{F}}^*_{k+1}| \geqq |\hat{\mathscr{F}}^*_k|$ for $1 \leqq k \leqq s-1$. We have

$$(3) \qquad |\hat{\mathscr{F}}^*_{k+1}|-|\hat{\mathscr{F}}^*_k| = (2^{k+1}-1)\,q_{k+1}-(2^k-1)\,q_k = 2^k(2q_{k+1}-q_k)+(q_k-q_{k+1}).$$

It is obvious that $q_k \geqq q_{k+1}$ so that if $2q_{k+1}-q_k \geqq 0$ we are through. We also have $q_{k+1}2^k+r_{k+1}=q_k 2^{k\pm 1}+r_k$ $(=n)$ so that $2^{k-1}(q_k-2q_{k+1})=r_{k+1}-r_k<2^k$ from which we get $q_k-2q_{k+1} \leqq 1$. There remains only the possibility that $q_k-2q_{k+1}=1$. We then get, from (3),

$$|\hat{\mathscr{F}}^*_{k+1}|-|\hat{\mathscr{F}}^*_k| = 2(r_k-r_{k+1})+q_{k+1}+1 =$$

$$= 2(r_k-r_{k+1})+\frac{n-r_{k+1}}{2^k}+1 =$$

$$= \frac{1}{2^k}\left(n+2^{k+1}r_k-(2^{k+1}+1)\,r_{k+1}+2^k\right).$$

Now $r_{k+1}=r_k$ or $2^{k-1}+r_k$. Thus

$$|\hat{\mathscr{F}}^*_{k+1}|-|\hat{\mathscr{F}}_k^*| \geqq \frac{1}{2^k}(n-2^{2k}-r_k+2^{k-1}) > \frac{1}{2^k}(n-2^{2k}) > 0,$$

since $n \geqq 5\cdot 2^{2s-3}$. Thus (iv) is satisfied.

In order for (v) to hold we must have

$$(2^{s-1}-1)\,q_{s-1}+r_{s-1}+r_{s-2}+\ldots+r_{t+1} \leqq (2^s-1)\,q_s.$$

This inequality is equivalent to

$$(2^{s-1}-1)\left(\frac{n-r_{s-1}}{2^{s-2}}\right)+r_{s-1}+r_{s-2}+\ldots+r_{t+1} \leqq (2^s-1)\left(\frac{n-r_s}{2^{s-1}}\right)$$

and this in turn to

$$n \geqq 2^{s-1}(r_{s-2}+\ldots+r_{t+1})+(2-2^{s-1})\,r_{s-1}+(2^s-1)\,r_s. \tag{4}$$

The right side of (4) assumes its largest value when $t=1$, $r_{s-1}=1$ and $r_k=2^{k-1}-1$ for $k\neq s-1$. It is then easy to check that, if $n\geqq 5\cdot 2^{2s-3}$, (4), and hence also (v), is satisfied. It now follows that the family $\mathscr{F}$ manufactured by the tower construction does not have property $\mathrm{B}(s)$. Since

$$|\mathscr{F}| \leqq \sum_{k=1}^{s}(2^{k+1}-2) = 2^{s+2}-2s-4,$$

(2) holds for all $s\geqq 1$, $n\geqq 5\cdot 2^{2s-3}$. ∎

We conclude with some remarks.

It may be the case that for every $s\geqq 4$, the inequality $m(n,s)<m(s)$ holds for some n. For example, Aizley and Selfridge [3] have announced that $m(4)\geqq 19$ and Theorem 1 gives $m(8,4)\leqq 15$. However, we do not know whether $m(n,5)<m(5)$ for any n.

The choice $\mathscr{F}_k=\mathscr{V}_k\cup\mathscr{V}_k'$ in the proof of Theorem 2 may not be the most appropriate one, in the sense that a different choice may yield a better upper bound for $m(n,s)$. It should be noted, however, that it would not suffice to choose $\mathscr{F}_k=\mathscr{V}_k$ since, for $k\geqq 4$, there are $(k-1)$-subsets of $\hat{\mathscr{V}}_k$ which are not in the complement of any hyperplane. Nor would it suffice to choose $\mathscr{F}_k=\mathscr{V}_k'$. $\mathscr{V}_k'$ has the property that every $(k-1)$-subset of $\hat{\mathscr{V}}_k'$ occurs in some member, but, unfortunately, it has property $\mathrm{B}(k)$. We are presently investigating other candidates and hope to be able to report on these investigations in a subsequent paper.

It would not suffice in constructing $\mathscr{F}_k$ to take $\mathscr{V}_k'=\{H\colon H\in\mathscr{H}_k\}$ because then $|\hat{\mathscr{F}}_k|=2^k$ and (iv) and (v) would fail to hold. This is the reason for trading 0 for h.

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